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THE LATTICE VERTEX OPERATOR ALGEBRA $V_{\sqrt{2}D_l}$ AND SOME VERTEX OPERATOR ALGEBRAS CONSTRUCTED FROM $\mathbb{Z}_8$-CODES

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In this note, we shall discuss a construction of vertex operator algebra from $\mathbb{Z}_8$-codes and the lattice vertex operator algebra $V_{\sqrt{2}D_l}$. This construction is essentially a commutant or coset construction associated with certain lattice VOAs constructed from the lattice $V_{\sqrt{2}D_l}$. Most of the materials are already written in [3, 4, 9]. Please refer to the corresponding references for more details.

1. A GLUE LATTICE ASSOCIATED WITH $\sqrt{2}D_l$

We shall start by constructing some glue lattice L_D from a $\mathbb{Z}_8$-code. First, let

$$D_l = \left\{ (x_1,\dots,x_n) \in \mathbb{Z}^l \,\middle|\, \sum_{i=1}^{l} x_i \text{ is even} \right\}, \quad l = 3,4,\dots,$$

be the root lattice of type D_l. Then the dual lattice of D_l is

$$\begin{aligned} D_l^* &= \left\{ y \in \mathbb{Q}\otimes_{\mathbb{Z}} D_l \,\middle|\, \langle x,y\rangle = \sum_{i=1}^{l} x_i y_i \in \mathbb{Z} \text{ for all } x \in D_l \right\} \\ &= \left\{ \frac{1}{2}(y_1,\dots,y_n) \,\middle|\, \text{all } y_i\text{'s are integers and have the same parity} \right\}. \end{aligned}$$

Note that $D_l^*/D_l \cong \mathbb{Z}_4$ if l is odd and $D_l^*/D_l \cong \mathbb{Z}_2$ if l is even

Let L be a lattice with basis $\{\alpha_1,\alpha_2,\dots,\alpha_l\}$ such that $\langle \alpha_i,\alpha_j\rangle = 2\delta_{ij}$ and $N = \sum_{i,j=1}^{l} \mathbb{Z}(\alpha_i \pm \alpha_j)$. Then, L is isomorphic to a direct sum of l copies of the root lattice of type A_1 and $N \cong \sqrt{2}D_l = \left\{ \sum_{i=1}^{l} a_i\alpha_i \middle| \sum_{i=1}^{l} a_i \equiv 0 \mod 2 \right\}$. Moreover, the dual lattice of N is

$$N^* \cong \frac{1}{\sqrt{2}} D_l^* = \left\{ \frac{1}{4}\sum_{i=1}^{l} b_i\alpha_i \,\middle|\, \text{all } b_i\text{'s are integers and have the same parity} \right\}.$$

Note that $N^*/N \cong (\mathbb{Z}_2)^{l-1} \times \mathbb{Z}_8$ when l is odd.

Actually, if we set $\gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_l$. Then, γ is a vector of square length $2l$ and the subgroup generated by the coset $\frac{\gamma}{4} + N$ is a cyclic group of order 8 in N^*/N.

From now on, we shall always assume l is odd and $N \cong \sqrt{2}D_l$. First, let us consider the sublattice R generated by the following eight cosets of N in L

$$\begin{array}{llll} N, & \dfrac{\gamma}{4} + N, & \dfrac{\gamma}{2} + N, & \dfrac{3\gamma}{4} + N, \\ \gamma + N, & \dfrac{5\gamma}{4} + N, & \dfrac{3\gamma}{2} + N, & \dfrac{7\gamma}{4} + N. \end{array}$$

Then we have

$$R/N =< \frac{\gamma}{4} + N >\cong \mathbb{Z}_8.$$

For simplicity, we shall denote $L^i = \frac{i}{4}\gamma + N$ for any $i \in \mathbb{Z}_8$.

Let $R^n = R \oplus \cdots \oplus R$ be the orthogonal sum of n copies of R. For any $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{Z}_8^n$, we define

$$L_\delta = L^{\delta_1} + \cdots + L^{\delta_n} = \left\{(x_1, \ldots, x_n) \in R^n \mid x_i \in L^{\delta_i}, i = 0, \ldots, 7\right\}.$$

For any subset $D \subset \mathbb{Z}_8^n$, we define

$$L_D = \bigcup_{\delta \in D} L_\delta.$$

Definition 1.1. Let $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{Z}_8^n$. The Euclidean weight of δ is defined to be

$$\mathrm{wt}\ \delta = \sum_{i=1}^{n} \min\{{\delta_i}^2, (8 - \delta_i)^2\} \in \mathbb{Z},$$

where $\delta_i \in \{0, 1, \ldots, 7\}$ are considered as integers.

Definition 1.2. A linear $\mathbb{Z}_8$ code D is said to be *doubly even* if

$$\mathrm{wt}\ \delta \equiv 0 \mod 16$$

for any $\delta = (\delta_1, \ldots, \delta_n) \in D$.

Theorem 1.3. *Let $D \subset \mathbb{Z}_8^n$ be a doubly even $\mathbb{Z}_8$-code. Then L_D is an even lattice.*

Proof. Since D is a linear code, it is clear that L_D is closed under addition. Thus, it is a lattice.

For any $a \in L^i$, $i \in \mathbb{Z}_8$,

$$\langle a, a\rangle \equiv \langle \frac{i\gamma}{4}, \frac{i\gamma}{4}\rangle \equiv \frac{l}{8} i^2 \equiv \frac{l}{8}\min\{i^2, (8-i)^2\} \mod 2.$$

Thus, for any $\delta \in D$ and $x = (x_1, \dots, x_n) \in L_\delta$,

$$\langle x, x\rangle = \sum_{i=1}^{n} \langle x_i, x_i\rangle \equiv \frac{l}{8}\sum_{i=1}^{n}\min\{\delta_i{}^2, (8-\delta_i)^2\} \equiv 0 \mod 2.$$

Hence, L_D is even. □

Corollary 1.4. *Let D be a doubly even $\mathbb{Z}_8$ code. Then the Fock space $V_{L_D} = S(\hat{\mathfrak{h}}_{\mathbb{Z}}^-) \otimes \mathbb{C}\{L_D\}$ is a vertex operator algebra. Moreover,*

$$V_{L_D} = \bigoplus_{\delta \in D} \left(\bigotimes_{i=1}^{n} V_{L^{\delta_i}} \right) \qquad \textit{as a vector space.}$$

2. Conformal vectors

In this section, we shall recall the construction of certain conformal vectors in $V_{\sqrt{2}D_l}$ [4].

Let

$$N = \sum_{i,j=1}^{l} \mathbb{Z}(\alpha_i \pm \alpha_j)$$

be a sublattice of L, which is isomorphic to the root lattice of type $\sqrt{2}D_l$. We choose the following elements as the simple roots of type D_l:

$$\beta_1 = (\alpha_1 + \alpha_2)/\sqrt{2}, \qquad \beta_2 = (-\alpha_2 + \alpha_3)/\sqrt{2}, \qquad \beta_3 = (-\alpha_1 + \alpha_2)/\sqrt{2},$$

$$\beta_i = (-\alpha_i + \alpha_{i+1})/\sqrt{2} \qquad \text{for} \quad 3 \le i \le l-1.$$

$$\Phi_l^+ = \{(\alpha_i + \alpha_j)/\sqrt{2},\ (-\alpha_i + \alpha_j)/\sqrt{2} \,|\, 1 \le i < j \le l\}$$

is the set of positive roots. Let

$$w^{\pm}(\beta) = \frac{1}{2}\beta(-1)^2 \pm (e^{\sqrt{2}\beta} + e^{-\sqrt{2}\beta}) \tag{2.1}$$

and set

(2.2)

$$\begin{aligned}
s^1 &= \frac{1}{4} w^-(\beta_1), \\
s^2 &= \frac{1}{5}\left(w^-(\beta_1) + w^-(\beta_2) + w^-(\beta_1 + \beta_2)\right), \\
s^r &= \frac{1}{2r} \sum_{1 \le i < j \le r} \left(w^-\left((\alpha_i + \alpha_j)/\sqrt{2}\right) + w^-\left((-\alpha_i + \alpha_j)/\sqrt{2}\right)\right), \quad 3 \le r \le l, \\
\omega &= \frac{1}{4(l-1)} \sum_{\beta \in \Phi_l^+} \beta(-1)^2.
\end{aligned}$$

It was shown by [5] that the elements

$$\omega^1 = s^1, \qquad \omega^i = s^i - s^{i-1}, 2 \le i \le l, \qquad \omega^{l+1} = \omega - s^l \tag{2.3}$$

are mutually orthogonal conformal vectors. Their central charges $c(\omega^i)$ are as follows:

$$c(\omega^1) = 1/2, \quad c(\omega^2) = 7/10, \quad c(\omega^3) = 4/5, \quad \text{and} \quad c(\omega^i) = 1 \text{ for } 4 \le i \le l{+}1.$$

The subalgebra $\mathrm{Vir}(\omega^i)$ of the vertex operator algebra V_N generated by ω^i is isomorphic to the Virasoro vertex operator algebra $L(c(\omega^i), 0)$ which is the irreducible highest weight module for the Virasoro algebra with central charge $c(\omega^i)$ and highest weight 0. Moreover, the subalgebra T of V_N generated by these conformal vectors is a tensor product of $\mathrm{Vir}(\omega^i)$'s, namely,

$$\begin{aligned}
T &= \mathrm{Vir}(\omega^1) \otimes \cdots \otimes \mathrm{Vir}(\omega^{l+1}) \\
&\cong L(c(\omega^1), 0) \otimes \cdots \otimes L(c(\omega^{l+1}), 0)
\end{aligned}$$

and V_N is completely reducible as a T-module.

Next, we shall consider three automorphisms θ_1, θ_2, σ of order two of the vertex operator algebra $V_{\mathbb{Z}\alpha}$ associated with a rank one lattice $\mathbb{Z}\alpha$, where $\langle\alpha,\alpha\rangle = 2$ (cf. [3, 4]). They are determined by

$$\begin{aligned}
&\theta_1 : \alpha(-1) \longmapsto \alpha(-1), && e^{\alpha} \longmapsto -e^{\alpha}, && e^{-\alpha} \longmapsto -e^{-\alpha},\\
&\theta_2 : \alpha(-1) \longmapsto -\alpha(-1), && e^{\alpha} \longmapsto e^{-\alpha}, && e^{-\alpha} \longmapsto e^{\alpha},\\
&\sigma : \alpha(-1) \longmapsto e^{\alpha}+e^{-\alpha}, && e^{\alpha}+e^{-\alpha} \longmapsto \alpha(-1), && e^{\alpha}-e^{-\alpha} \longmapsto -(e^{\alpha}-e^{-\alpha}).
\end{aligned}$$

The automorphism θ_1 maps $u \otimes e^{\beta}$ to $(-1)^{\langle\alpha,\beta\rangle/2} u \otimes e^{\beta}$ for $u \in M(1)$ and $\beta \in \mathbb{Z}\alpha$ and θ_2 is the automorphism induced from the isometry $\beta \longmapsto -\beta$ of $\mathbb{Z}\alpha$. Note also that

$$\sigma\theta_1\sigma = \theta_2, \quad \sigma(\alpha(-1)^2) = \alpha(-1)^2 \quad \text{and} \quad \sigma(e^{\pm\alpha}) = (\alpha(-1) \mp (e^{\alpha} - e^{-\alpha}))/2.$$

Let L be a lattice with basis $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ such that $\langle\alpha_i, \alpha_j\rangle = 2\delta_{ij}$. Then, the vertex operator algebra V_L is a tensor product $V_L = V_{\mathbb{Z}\alpha_1} \otimes \cdots \otimes V_{\mathbb{Z}\alpha_l}$ of $V_{\mathbb{Z}\alpha_i}$'s. Using the automorphisms θ_1, θ_2, and σ of $V_{\mathbb{Z}\alpha_i}$ described above, we can define three automorphisms ψ_1, ψ_2, and τ of V_L of order two by

$$\psi_1 = \theta_1 \otimes \cdots \otimes \theta_1, \qquad \psi_2 = \theta_2 \otimes \cdots \otimes \theta_2, \qquad \tau = \sigma \otimes \cdots \otimes \sigma.$$

Then

$$\psi_1(u \otimes e^{\beta}) = (-1)^{\langle\alpha_1+\alpha_2+\cdots+\alpha_l,\beta\rangle/2} u \otimes e^{\beta} \tag{2.4}$$

for $u \in M(1)$ and $\beta \in L$, ψ_2 is the automorphism induced from the isometry $\beta \longmapsto -\beta$ of L, and $\tau\psi_1\tau = \psi_2$.

Let $\varphi : V_L \to V_L$ be an automorphism defined by

$$\varphi : u \otimes e^{\beta} \longmapsto (-1)^{\langle\alpha_2+\alpha_3,\beta\rangle/2} u \otimes e^{\beta},$$

where $u \in M(1)$ and $\beta \in L$. The automorphism φ acts as θ_2 on $V_{\mathbb{Z}\alpha_2}$ and $V_{\mathbb{Z}\alpha_3}$ and acts as the identity on $V_{\mathbb{Z}\alpha_i}$ for $i \neq 2$, 3. Set $\rho = \varphi\tau$. Then we have

Lemma 2.1 (Dong at. el. [4]).

$$\rho(s^1) = \frac{1}{4}w^-(\beta_3), \qquad \rho(s^2) = \frac{1}{5}(w^-(\beta_3) + w^-(\beta_2) + w^-(\beta_2 + \beta_3)),$$
$$\rho(s^r) = \tau(s^r), \quad 3 \le r \le l, \quad \rho(\omega) = \omega.$$

Let $\widetilde{\omega}^i = \rho(\omega^i)$ and set

$$\gamma_r = \alpha_1 + \alpha_2 + \cdots + \alpha_r - r\alpha_{r+1}, \quad 1 \le r \le l-1, \tag{2.5}$$
$$\gamma_l = \alpha_1 + \alpha_2 + \cdots + \alpha_l.$$

Lemma 2.2 (cf. [4]). (1) *The vectors* $\widetilde{\omega}^1$, $\widetilde{\omega}^2$, *and* $\widetilde{\omega}^3$ *are the mutually orthogonal conformal vectors of* $V_{\mathbb{Z}(\alpha_1-\alpha_2)+\mathbb{Z}(\alpha_2-\alpha_3)} \cong V_{\sqrt{2}A_2}$ *defined in* [5].

(2) $\widetilde{\omega}^{r+1} = \dfrac{1}{4r(r+1)}\gamma_r(-1)^2$ *for* $3 \le r \le l-1$.

(3) $\widetilde{\omega}^{l+1} = \dfrac{1}{4l}\gamma_l(-1)^2$.

Note that for $3 \le r \le l$, the element $\widetilde{\omega}^{r+1}$ is the Virasoro element of the vertex operator algebra $V_{\mathbb{Z}\gamma_r}$ associated with a rank one lattice $\mathbb{Z}\gamma_r$.

Set $U^{\pm} = \{v \in U \mid \psi_2(v) = \pm v\}$ for any ψ_2-invariant subspace U of V_L.

Lemma 2.3 (cf. [4]). (1) $N = \{\beta \in L \mid \langle \alpha_1 + \cdots + \alpha_l, \beta\rangle \equiv 0 \pmod 4\}$.

(2) $V_N = \{v \in V_L \mid \psi_1(v) = v\}$.

(3) $\rho(V_N) = V_L^+$.

The last assertion of the above lemma implies that the decomposition of V_N into a direct sum of irreducible T-modules is equivalent to that of V_L^+ as a $\widetilde{T}$-module, where $\widetilde{T} = \rho(T)$ is of the form

$$\widetilde{T} = \mathrm{Vir}(\widetilde{\omega}^1) \otimes \cdots \otimes \mathrm{Vir}(\widetilde{\omega}^{l+1})$$
$$\cong L(\tfrac{1}{2},0) \otimes L(\tfrac{7}{10},0) \otimes L(\tfrac{4}{5},0) \otimes L(1,0) \otimes \cdots \otimes L(1,0).$$

Next, we shall study the decomposition of $V_N \cong V_L^+$. Details are again written in [3, 4]. As in [3, 4], we set

$$E = \mathbb{Z}(\alpha_1 - \alpha_2) + \mathbb{Z}(\alpha_2 - \alpha_3) \qquad \text{and} \qquad D = E + \mathbb{Z}\gamma_3 + \cdots + \mathbb{Z}\gamma_l.$$

The elements $\alpha_1 - \alpha_2$, $\alpha_2 - \alpha_3$, $\gamma_3, \ldots, \gamma_l$ form a basis of the lattice D. Since $\gamma_r = \sum_{i=1}^r i(\alpha_i - \alpha_{i+1})$, we can take

$$\{\alpha_1 - \alpha_2,\ \alpha_2 - \alpha_3,\ 3(\alpha_3 - \alpha_4),\ \ldots,\ (l-2)(\alpha_{l-2} - \alpha_{l-1}),\ \gamma_{l-1},\ \gamma_l\}$$

as another basis. The lattices E, $\mathbb{Z}\gamma_3, \ldots, \mathbb{Z}\gamma_l$ are mutually orthogonal, so the vertex operator algebra V_D associated with the lattice D is a tensor product $V_D = V_E \otimes V_{\mathbb{Z}\gamma_3} \otimes \cdots \otimes V_{\mathbb{Z}\gamma_l}$.

Next, we want to describe the cosets of D in L. Set

$$\xi_r = \frac{1}{r(r+1)}\gamma_r,\ 1 \le r \le l-1, \qquad \text{and} \qquad \xi_l = \frac{1}{l}\gamma_l.$$

Then we have

$$-\xi_1 + \xi_2 = \frac{1}{3}(-(\alpha_1 - \alpha_2) + (\alpha_2 - \alpha_3)). \tag{2.6}$$

and

$$-\xi_1 + \xi_2 + \cdots + \xi_l = \alpha_2. \tag{2.7}$$

To simplify the notation, we set $\eta = -\xi_1 + \xi_2$.

Lemma 2.4. *$|D + \mathbb{Z}\alpha_2 : D|$ is equal to the least common multiple of* 3, 4, ..., *l.*

Note that $D + \mathbb{Z}\alpha_2 = L$ for $3 \le l \le 5$. Indeed, the coset $D + \alpha_2$ contains α_1, α_2, and α_3. Moreover, $\alpha_4 \in D + 9\alpha_2$ if $l = 4$, and $\alpha_4 \in D + 21\alpha_2$ and $\alpha_5 \in D + 36\alpha_2$ if $l = 5$. Hence $n\alpha_2$, $0 \le n \le d-1$, where d denotes the least common multiple of 3, 4, ..., l, form a complete system of representatives of the cosets of D in L in these three cases. However, $D + \mathbb{Z}\alpha_2 \ne L$ for $l \ge 6$.

We shall use the following elements to describe all the cosets of D in L. For $m_3, \ldots, m_{l-2}, n \in \mathbb{Z}$ we let

$$
\begin{aligned}
\lambda &= \lambda(m_3, m_4, \ldots, m_{l-2}, n) \\
&= m_3(\alpha_3 - \alpha_4) + m_4(\alpha_4 - \alpha_5) + \cdots + m_{l-2}(\alpha_{l-2} - \alpha_{l-1}) + n\alpha_2 \\
&\equiv (m_3 + n)\eta + \sum_{r=3}^{l-3}((r+1)m_r - rm_{r+1} + n)\xi_r \\
&\qquad + ((l-1)m_{l-2} + n)\xi_{l-2} + n\xi_{l-1} + n\xi_l \qquad (\mathrm{mod}\ D).
\end{aligned} \tag{2.8}
$$

The last congruence modulo D comes from (4.1), (4.2), and the fact that $\alpha_r - \alpha_{r+1} = -(r-1)\xi_{r-1} + (r+1)\xi_r$.

Lemma 2.5. (1) $\{\lambda = \lambda(m_3, \ldots, m_{l-2}, n) \mid 0 \le m_r \le r-1, 0 \le n \le l(l-1) - 1\}$ *forms a complete system of representatives of the cosets of D in L.*
(2) *Every element in the coset $D + \lambda$ can be uniquely written in the form*

$$
\begin{aligned}
&(\nu + (m_3 + n)\eta) + \sum_{i=3}^{l-3}(\mu_i + ((i+1)m_i - im_{i+1} + n)\xi_i) \\
&\qquad + (\mu_{l-2} + ((l-1)m_{l-2} + n)\xi_{l-2}) + (\mu_{l-1} + n\xi_{l-1}) + (\mu_l + n\xi_l)
\end{aligned}
$$

for $\nu \in E$ and $\mu_i \in \mathbb{Z}\gamma_i$.

Lemma 2.6. *For $\lambda = \lambda(m_3, \ldots, m_{l-2}, n)$, $0 \le m_r \le r-1$, $0 \le n \le l(l-1)-1$, we have $D + \lambda = D - \lambda$ if and only if $m_3, \ldots, m_{l-2}$, and n satisfy one of the following conditions.*

(1) $n = 0$, *and $m_r = 0$ if r is odd and $m_r = 0$ or $r/2$ if r is even.*
(2) $n = l(l-1)/2$ *and $2m_r + l(l-1) \equiv 0 \pmod r$. Such an m_r is unique if r is odd and there are exactly two such m_r if r is even.*

The automorphism ψ_2 fixes the conformal vectors $\widetilde{\omega}^1, \ldots, \widetilde{\omega}^{l+1}$, and so $\widetilde{T} \subset V_D^+$. In particular, ψ_2 is a $\widetilde{T}$-module isomorphism. We have $\psi_2(V_{D+\lambda}) = V_{D-\lambda}$, and thus $V_{D-\lambda}$ is isomorphic to $V_{D+\lambda}$ as a $\widetilde{T}$-module. If $D+\lambda \ne D-\lambda$, the fixed point subspace $(V_{D+\lambda}\oplus V_{D-\lambda})^+$ in $V_{D+\lambda}\oplus V_{D-\lambda}$ is equal to $\{v+\psi_2(v)\mid v\in V_{D+\lambda}\}$ and it is isomorphic to $V_{D+\lambda}$.

If $D+\lambda=D-\lambda$ for $\lambda=\lambda(m_3,\dots,m_{l-2},n)$, then $m_3,\ \dots,\ m_{l-2}$, and n satisfy the conditions in Lemma 4.3. In this case (4.5) is in the following form:

$$V_{D+\lambda}=V_E\otimes V_{\mathbb{Z}\gamma_3+b_3\gamma_3}\otimes\cdots\otimes V_{\mathbb{Z}\gamma_l+b_l\gamma_l},$$

with $b_i\in\{0,\frac{1}{2}\}$ for $3\le i\le l-1$ and $b_l=0$ or $b_l\in\{0,\frac{1}{2}\}$ depending on whether l is odd or even.

For $\varepsilon=(\varepsilon_0,\varepsilon_3,\dots,\varepsilon_l)$ with $\varepsilon_i=+$ or $-$, set

$$V^{\varepsilon}_{D+\lambda}=V^{\varepsilon_0}_E\otimes V^{\varepsilon_3}_{\mathbb{Z}\gamma_3+b_3\gamma_3}\otimes\cdots\otimes V^{\varepsilon_l}_{\mathbb{Z}\gamma_l+b_l\gamma_l}. \tag{2.9}$$

Then

$$V^{+}_{D+\lambda}=\oplus_{\varepsilon}V^{\varepsilon}_{D+\lambda}, \tag{2.10}$$

where ε runs over all $\varepsilon=(\varepsilon_0,\varepsilon_3,\dots,\varepsilon_l)$ such that even number of ε_i's are $-$.

We divide a complete system of representatives of the cosets of D in L into three subsets Λ_1, Λ_2, and $-\Lambda_2$ so that $D+\lambda=D-\lambda$ if and only if $\lambda\in\Lambda_1$. Then

$$V_L=\Big(\oplus_{\lambda\in\Lambda_1}V_{D+\lambda}\Big)\oplus\Big(\oplus_{\lambda\in\Lambda_2}V_{D+\lambda}\Big)\oplus\Big(\oplus_{\lambda\in\Lambda_2}V_{D-\lambda}\Big).$$

By the above argument, we conclude that

Theorem 2.7 (Dong at.el. [4]). *As $\widetilde{T}$-modules,*

$$V^+_L\cong\Big(\oplus_{\lambda\in\Lambda_1}V^+_{D+\lambda}\Big)\oplus\Big(\oplus_{\lambda\in\Lambda_2}V_{D+\lambda}\Big).$$

Furthermore, the decomposition of $V^+_{D+\lambda}$, $\lambda\in\Lambda_1$, and $V_{D+\lambda}$, $\lambda\in\Lambda_2$, into a direct sum of irreducible $\widetilde{T}$-modules is given by (4.5) through (4.10) and (5.1) through (5.7).

By the same method, we also have the decomposition of $V^-_L\cong(\oplus_{\lambda\in\Lambda_1}V^-_{D+\lambda})\oplus(\oplus_{\lambda\in\Lambda_2}V_{D+\lambda})$ into a direct sum of irreducible $\widetilde{T}$-modules.

3. Coset construction of vertex operator algebra

In this section, we shall use the decomposition obtained in the last section and the coset construction to construct some vertex operator algebras. First, we shall recall the definition of commutant (or coset) subalgebras of a vertex operator algebra (cf. [7]).

Definition 3.1. Let $(V, Y, \omega, 1)$ be a vertex operator algebra and $(W, Y, \omega', 1)$ be a vertex operator subalgebra of V. Note that the Virasoro elements of V and W are different. The commutant of W in V is defined to be the subspace

$$W^c = \{v \in V \,|\, w_n v = 0, \text{ for all } w \in W \text{ and } n \geq 0\}$$

Similarly, for any V-module M, the commutant of W in M is defined to be

$$M^c = \{u \in M \,|\, w_n u = 0, \text{ for all } w \in W \text{ and } n \geq 0\}$$

The following facts are well-known in the theory of vertex operator algebra (cf. [7], for example).

Proposition 3.2. *$(W^c, Y, \omega'', 1)$ is a vertex operator algebra where $\omega'' = \omega - \omega'$ and M^c is a W^c-module for any V-module M.*

Now, let $L^i = \frac{i}{4}\gamma + N$ be defined as in Section 1. Denote

$$M^i = \left\{v \in V_{L^i} \,|\, (\omega^1)_1 v = (\omega^2)_1 v = \cdots = (\omega^l)_1 v = 0\right\},$$

where ω^i, $i = 0, \ldots, 7$, are defined as in (2.3). Note that M^0 is a VOA and M^i, $i = 0, \ldots, 7$, are M^0-modules.

For any $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{Z}_8^n$, we define

$$M_\delta = \bigotimes_{i=1}^{n} M^{\delta_i}.$$

For any $\mathbb{Z}_8$ code D, define

$$M_D = \bigoplus_{\delta \in D} M_\delta.$$

Theorem 3.3. *If D is a doubly even $\mathbb{Z}_8$ code, then M_D is a vertex operator algebra.*

Proof. Let D be a doubly even $\mathbb{Z}_8$ code. Then L_D is an even lattice and

$$V_{L_D} = \bigoplus_{\delta \in D} (V_{L^{\delta_1}} \otimes \cdots \otimes V_{L^{\delta_n}})$$

is a VOA (cf. Section 1). Note that

$$M_D = \left\{ v \in V_{L_D} \mid (\hat{\omega}^1)_1 v = (\hat{\omega}^2)_1 v = \cdots (\hat{\omega}^l) v = 0 \right\},$$

where $\hat{\omega}^i = \omega^i \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \omega^i \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes \omega^i$, $i = 1, \ldots, l$.

Thus, M_D is a vertex operator subalgebra of V_{L_D}. □

Remark 3.4. As in [8, 11], one can define the so-called coordinate automorphisms for M_D as follows:

For any $\alpha \in (\mathbb{Z}_8^*)^n$,

$$\sigma_\alpha(u) = \xi^{(\alpha,\beta)} u \qquad \text{for } u \in M_\beta,$$

where ξ is a primitive 8-th root of unity and $\mathbb{Z}_8^* = \{1, 3, 5, 7\}$ is the units of the ring $\mathbb{Z}_8$.

4. The case for $l = 3$

In this section, we shall explain the above construction for the case $l = 3$ in more details. The other cases, in principle, can be done in a similar way.

Let $L = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3$ and $N = \sum_{i,j=1}^{3} \mathbb{Z}(\alpha_i \pm \alpha_j)$. Moreover, we shall denote $E = \mathbb{Z}(\alpha_1 - \alpha_2) \oplus \mathbb{Z}(\alpha_2 - \alpha_3)$ and $F = \mathbb{Z}\gamma$, where $\gamma = \alpha_1 + \alpha_2 + \alpha_3$.

Theorem 4.1 (Kitazume at.el. [9]). *Let*

$$N = \sum_{i,j=1}^{3} \mathbb{Z}(\alpha_i \pm \alpha_j) \cong \sqrt{2}D_3 \cong \sqrt{2}A_3.$$

Then

$$\begin{aligned}
V_N &\cong (V_E^+ \otimes V_F^+) \oplus (V_E^- \otimes V_F^-) \oplus \left(V_{E+\sqrt{2}(\beta_1-\beta_2)/3} \otimes V_{\frac{\gamma}{3}+F}\right), \\
V_{\frac{1}{4}\gamma+N} &\cong \left(V_E^{\mathcal{T},-} \otimes V_F^{T_1,-}\right) \oplus \left(V_E^{\mathcal{T},+} \otimes V_F^{T_1,+}\right), \\
V_{\frac{1}{2}\gamma+N} &\cong \left(V_E^+ \otimes V_{\frac{\gamma}{2}+F}^+\right) \oplus \left(V_E^- \otimes V_{\frac{\gamma}{2}+F}^-\right) \oplus \left(V_{E+\sqrt{2}(\beta_1-\beta_2)/3} \otimes V_{\frac{\gamma}{6}+F}\right), \\
V_{\frac{3}{4}\gamma+N} &\cong \left(V_E^{\mathcal{T},-} \otimes V_F^{T_2,+}\right) \oplus \left(V_E^{\mathcal{T},+} \otimes V_F^{T_2,-}\right), \\
V_{\gamma+N} &\cong (V_E^+ \otimes V_F^-) \oplus (V_E^- \otimes V_F^+) \oplus \left(V_{E+\sqrt{2}(\beta_1-\beta_2)/3} \otimes V_{\frac{\gamma}{3}+F}\right), \\
V_{\frac{5}{4}\gamma+N} &\cong \left(V_E^{\mathcal{T},-} \otimes V_F^{T_1,+}\right) \oplus \left(V_E^{\mathcal{T},+} \otimes V_F^{T_1,-}\right), \\
V_{\frac{3}{2}\gamma+N} &\cong \left(V_E^+ \otimes V_{\frac{\gamma}{2}+F}^-\right) \oplus \left(V_E^- \otimes V_{\frac{\gamma}{2}+F}^+\right) \oplus \left(V_{E+\sqrt{2}(\beta_1-\beta_2)/3} \otimes V_{\frac{\gamma}{6}+F}\right), \\
V_{\frac{7}{4}\gamma+N} &\cong \left(V_E^{\mathcal{T},-} \otimes V_F^{T_2,-}\right) \oplus \left(V_E^{\mathcal{T},+} \otimes V_F^{T_2,+}\right),
\end{aligned}$$

where $V_E^{\mathcal{T}} = S(\hat{\mathfrak{h}}^-_{\mathbb{Z}+\frac{1}{2}}) \otimes \mathcal{T}$ is a ψ_2-twisted module of V_E and $\mathcal{T}$ is an irreducible $\widehat{E}/K$ module such that $e^a \cdot t = t$ for $a \in E$, $t \in \mathcal{T}$, and $K = \{\pm e^b | \, b \in 2E\}$ is a central extension of $2E$, and $V_F^{T_1}$ and $V_F^{T_2}$ are the two inequivalent irreducible ψ_2-twisted modules for V_F.

Theorem 4.2 (cf. [9]). *Let*

$$\begin{aligned}
M^i &= \left\{v \in V_{L^i} \,|\, (\omega^1)_1 v = (\omega^2)_1 v = 0\right\}, \text{ and} \\
W^i &= \{v \in V_{L^i} \,|\, (\omega^1)_1 v = 0 \text{ and } (\omega^2)_1 v = \frac{3}{5} v\}.
\end{aligned}$$

Then, M^0 is a simple VOA and M^i and W^i are irreducible M^0-modules.

Moreover, by Theorem 4.1,

$$
\begin{aligned}
M^0 &= \left(L(\tfrac{4}{5},0)\otimes V_F^+\right)\oplus\left(L(\tfrac{4}{5},3)\otimes V_F^-\right)\oplus\left(L(\tfrac{4}{5},\tfrac{2}{3})\otimes V_{\frac{\gamma}{3}+F}\right),\\
M^1 &= \left(L(\tfrac{4}{5},\tfrac{1}{8})\otimes V_F^{T_1,-}\right)\oplus\left(L(\tfrac{4}{5},\tfrac{13}{8})\otimes V_F^{T_1,+}\right),\\
M^2 &= \left(L(\tfrac{4}{5},0)\otimes V_{\frac{\gamma}{2}+F}^+\right)\oplus\left(L(\tfrac{4}{5},3)\otimes V_{\frac{\gamma}{2}+F}^-\right)\oplus\left(L(\tfrac{4}{5},\tfrac{2}{3})\otimes V_{\frac{\gamma}{6}+F}\right),\\
M^3 &= \left(L(\tfrac{4}{5},\tfrac{1}{8})\otimes V_F^{T_2,+}\right)\oplus\left(L(\tfrac{4}{5},\tfrac{13}{8})\otimes V_F^{T_2,-}\right),\\
M^4 &= \left(L(\tfrac{4}{5},0)\otimes V_F^-\right)\oplus\left(L(\tfrac{4}{5},3)\otimes V_F^+\right)\oplus\left(L(\tfrac{4}{5},\tfrac{2}{3})\otimes V_{\frac{\gamma}{3}+F}\right),\\
M^5 &= \left(L(\tfrac{4}{5},\tfrac{1}{8})\otimes V_F^{T_1,+}\right)\oplus\left(L(\tfrac{4}{5},\tfrac{13}{8})\otimes V_F^{T_1,-}\right),\\
M^6 &= \left(L(\tfrac{4}{5},0)\otimes V_{\frac{\gamma}{2}+F}^-\right)\oplus\left(L(\tfrac{4}{5},3)\otimes V_{\frac{\gamma}{2}+F}^+\right)\oplus\left(L(\tfrac{4}{5},\tfrac{2}{3})\otimes V_{\frac{\gamma}{6}+F}\right),\\
M^7 &= \left(L(\tfrac{4}{5},\tfrac{1}{8})\otimes V_F^{T_2,-}\right)\oplus\left(L(\tfrac{4}{5},\tfrac{13}{8})\otimes V_F^{T_2,+}\right),\\
W^0 &= \left(L(\tfrac{4}{5},\tfrac{7}{5})\otimes V_F^+\right)\oplus\left(L(\tfrac{4}{5},\tfrac{2}{5})\otimes V_F^-\right)\oplus\left(L(\tfrac{4}{5},\tfrac{1}{15})\otimes V_{\frac{\gamma}{3}+F}\right),\\
W^1 &= \left(L(\tfrac{4}{5},\tfrac{21}{40})\otimes V_F^{T_1,-}\right)\oplus\left(L(\tfrac{4}{5},\tfrac{1}{40})\otimes V_F^{T_1,+}\right),\\
W^2 &= \left(L(\tfrac{4}{5},\tfrac{7}{5})\otimes V_{\frac{\gamma}{2}+F}^+\right)\oplus\left(L(\tfrac{4}{5},\tfrac{2}{5})\otimes V_{\frac{\gamma}{2}+F}^-\right)\oplus\left(L(\tfrac{4}{5},\tfrac{1}{15})\otimes V_{\frac{\gamma}{6}+F}\right),\\
W^3 &= \left(L(\tfrac{4}{5},\tfrac{21}{40})\otimes V_F^{T_2,+}\right)\oplus\left(L(\tfrac{4}{5},\tfrac{1}{40})\otimes V_F^{T_2,-}\right),\\
W^4 &= \left(L(\tfrac{4}{5},\tfrac{7}{5})\otimes V_F^-\right)\oplus\left(L(\tfrac{4}{5},\tfrac{2}{5})\otimes V_F^+\right)\oplus\left(L(\tfrac{4}{5},\tfrac{1}{15})\otimes V_{\frac{\gamma}{3}+F}\right),\\
W^5 &= \left(L(\tfrac{4}{5},\tfrac{21}{40})\otimes V_F^{T_1,+}\right)\oplus\left(L(\tfrac{4}{5},\tfrac{1}{40})\otimes V_F^{T_1,-}\right),\\
W^6 &= \left(L(\tfrac{4}{5},\tfrac{7}{5})\otimes V_{\frac{\gamma}{2}+F}^-\right)\oplus\left(L(\tfrac{4}{5},\tfrac{2}{5})\otimes V_{\frac{\gamma}{2}+F}^+\right)\oplus\left(L(\tfrac{4}{5},\tfrac{1}{15})\otimes V_{\frac{\gamma}{6}+F}\right),\\
W^7 &= \left(L(\tfrac{4}{5},\tfrac{21}{40})\otimes V_F^{T_2,-}\right)\oplus\left(L(\tfrac{4}{5},\tfrac{1}{40})\otimes V_F^{T_2,+}\right)
\end{aligned}
$$

as $L(4/5,0) \otimes V_F^+$*-modules.*

Remark 4.3. In fact, one can show that M^i and W^i, $i = 0, \dots, 7$, are exactly all the inequivalent irreducible modules for M^0. Moreover, the fusion rules for M^0-modules are given as:

$$\begin{aligned} M^i \times M^j &= M^{i+j}, \\ M^i \times W^j &= W^{i+j}, \\ W^i \times W^j &= M^{i+j} + W^{i+j}, \end{aligned}$$

where $i, j \in \mathbb{Z}_8$.

Now, let us discuss the construction some irreducible M_D-modules using induced modules.

Let $U = U^{\delta_1} \otimes \cdots \otimes U^{\delta_n}$ be an irreducible $(M^0)^{\otimes n}$-module such that $U^{\delta_i} = M^{\delta_i}$ or W^{δ_i}, $\delta_i = 0, 1, \dots, 7$.

Define

$$\operatorname{Ind}^D U = \bigoplus_{\alpha \in D} \left(U^{\alpha_1 + \delta_1} \otimes \cdots \otimes U^{\alpha_n + \delta_n} \right),$$

where $U^{\alpha_i + \delta_i} = M^{\alpha_i + \delta_i}$ (or $W^{\alpha_i + \delta_i}$ respectively) if $U^{\delta_i} = M^{\delta_i}$ (or W^{δ_i} respectively). $\operatorname{Ind}^D U$ is called an induced module.

Theorem 4.4. *If* $(\delta, D) = 0$*, then* $\operatorname{Ind}^D U$ *is an* M_D*-module.*

Proof. First, we shall note that $U = U^{\delta_1} \otimes \cdots \otimes U^{\delta_n}$ can be considered as a subset of $V_{L_\delta} \cong \otimes_{i=1}^n V_{L^{\delta_i}}$ for any $\delta \in \mathbb{Z}_8^n$. Therefore,

$$\operatorname{Ind}^D U = \bigoplus_{\alpha \in D} \left(U^{\alpha_1 + \delta_1} \otimes \cdots \otimes U^{\alpha_n + \delta_n} \right) \subset V_{L_{\delta + D}}.$$

If $(\delta, D) = 0$, then $\langle L_D, L_{\delta+D} \rangle \subset \mathbb{Z}$. Thus, $V_{L_{\delta+D}}$ is a V_{L_D}-module. Note that M_D is a subVOA of V_{L_D} and the action of M_D on $\operatorname{Ind}^D U$ is closed. Thus, $\operatorname{Ind}^D U$ is a M_D-module. $\square$

Remark 4.5. If $(\delta, D) \neq 0$, we believe that $\mathrm{Ind}^D U$ will still define a g-twisted module of M_D, where g is an automorphism of M_D such that

$$g(u) = \xi^{(\delta,\alpha)}u, \qquad \text{for } u \in M_\alpha \text{ and } \alpha \in D,$$

and ξ is a primitive 8-th root of unity.

Remark 4.6. Suppose V_F^+ is rational. Then one can show that $\mathrm{Ind}^D U$ is an irreducible M_D-module. Moreover, all irreducible M_D-modules are induced modules and M_D is rational (cf. [9]).

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